

THE WEIL–PETERSSON GEOMETRY OF THE MODULI SPACE OF RIEMANN SURFACES

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ABSTRACT. In [4], Z. Huang showed that in the thick part of the moduli space \mathcal{M}_g of compact Riemann surfaces of genus g , the sectional curvature of the Weil–Petersson metric is bounded below by a constant depending on injectivity radius, but independent of the genus g . In this article, we prove this result by a different method. We also show that the same result holds for Ricci curvature. For the universal Teichmüller space equipped with Hilbert structure induced by Weil–Petersson metric, we prove that its sectional curvature is bounded below by a universal constant.

1. INTRODUCTION

There have been a lots of studies on the geometry of Weil–Petersson (WP) metric on moduli spaces of Riemann surfaces, especially regarding its curvature properties [1, 10, 16, 14, 11, 6, 9, 17, 18, 15, 7, 8, 3, 4]. In a pioneering work, Ahlfors [1] showed that the Ricci, holomorphic sectional and scalar curvatures of the WP metric are all negative. Later, Royden [10] showed that the holomorphic sectional curvature is bounded above by a negative constant, and he conjectured that on the moduli space \mathcal{M}_g of compact Riemann surfaces of genus g , this constant is equal to $\frac{-1}{2\pi(g-1)}$. By deriving more compact expressions for the Riemann tensors of the WP metric, Wolpert [16] verified Royden’s conjecture. He also showed that the Ricci curvature is bounded above by $\frac{-1}{2\pi(g-1)}$, and the scalar curvature is bounded above by $-\frac{3(3g-2)}{4\pi}$. In the communications between Wolpert and Tromba and between Wolpert and Royden, it was proved that the sectional curvature of the WP metric is also negative. Detailed proof of this result was given by Wolpert in [16] and by Tromba in [14]. Regarding the upper bound, it was proved in [3] that the sectional curvature does not have a negative upper bound.

Lower bounds of the curvatures are less considered. There are some results obtained by [11, 13, 5, 4]. The results of [11, 13, 5] showed that the sectional curvature is not bounded below on the moduli space \mathcal{M}_g . Therefore, attention must be shifted to find lower bounds of the sectional curvature on compact subsets of the moduli space \mathcal{M}_g . This problem was studied by Huang in [4]. To describe his result in more detail, we need to introduce some notions first. A point on the moduli space \mathcal{M}_g can be considered as a compact Riemann surface X of genus g endowed with a unique metric of constant curvature -1 , called the hyperbolic metric. The injectivity radius of X at a point $z \in X$, $\text{inj}(X; z)$, is defined as the supremum of all r for each the open set $U_z^r = \{w \in X : d(z, w) < r\}$ is isometric to a disc. The injectivity radius of X , $\text{inj}(X)$, is defined to be the infimum of $\text{inj}(X; z)$, $z \in X$.

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By a well-known result, $\text{inj}(X)$ is equal to one half of the length of the shortest closed geodesic of X . Given a positive constant r_0 , the thick part of moduli space \mathcal{M}_g (with respect to r_0), is defined as the subset of \mathcal{M}_g consisting of those points where the injectivity radius of the corresponding Riemann surfaces is greater than r_0 . In [4], Huang showed that on the thick part of the moduli space \mathcal{M}_g , the holomorphic sectional and sectional curvatures of the WP metric are both bounded below by negative constants $-C_1$ and $-C_2$ depending on r_0 , but independent of genus g . As a result, the Ricci and scalar curvatures are bounded below by $-C_3g$ and $-C_4g^2$ respectively, where C_3 and C_4 are two positive constants depending on r_0 , but independent of g . The main tool used by Huang is the analysis of harmonic maps between hyperbolic surfaces. In the present article, we are going to give a different proof of Huang's result without using harmonic maps. Moreover, we are going to improve the bounds $-C_3g$ and $-C_4g^2$ for Ricci and scalar curvatures to $-C_3$ and $-C_4g$ respectively. Explicit dependence of the constants C_1, C_2, C_3, C_4 on the injectivity radius r_0 is given.

In [12], we have defined a Hilbert structure on the universal Teichmüller space $T(1)$, so that the Weil–Petersson metric is a well-defined metric on $T(1)$. We have also obtained an explicit formula for the Riemann curvature tensor of the WP metric, which is generalization of the result of Wolpert [16]. In this article, we are going to show that the sectional curvature of the WP metric on $T(1)$ is bounded below by a universal negative constant. We also show that it does not have a negative upper bound.

The layout of this article is as follows. In Section 2, we review some necessary facts. In Section 3, we obtain the lower bounds of the curvatures of the WP metric on the moduli space \mathcal{M}_g as a function of injectivity radius. In Section 4, we find the lower bound of the sectional curvature of the WP metric on the universal Teichmüller space.

2. BACKGROUND

In this section, we present some necessary facts. Let $T(X)$ and $\mathcal{M}(X)$ be respectively the Teichmüller space and moduli space of a compact Riemann surface X of genus g , where $g \geq 2$. The Teichmüller space $T(X)$ has a complex analytic model described as follows. Let \mathbb{D} and \mathbb{D}^* be respectively the unit disc and its exterior. There is a Fuchsian group $\Gamma \in \text{PSU}(1, 1)$ such that the quotient of the unit disc \mathbb{D} by the action of Γ is X , i.e., $X \simeq \Gamma \backslash \mathbb{D}$. The space of bounded Beltrami differentials on X can be identified with the space of bounded Γ -automorphic $(-1, 1)$ differentials on \mathbb{D} , denoted by $\mathcal{A}^{-1,1}(\mathbb{D}, \Gamma)$, which consists of bounded functions μ on \mathbb{D} satisfying

$$\mu(\gamma(z)) \frac{\overline{\gamma'(z)}}{\gamma'(z)} = \mu(z).$$

Let $\mathcal{B}^{-1,1}(\mathbb{D}, \Gamma)$ be the unit ball of $\mathcal{A}^{-1,1}(\mathbb{D}, \Gamma)$ with respect to the sup-norm:

$$\|\mu\|_\infty = \sup_{z \in \mathbb{D}} |\mu(z)|.$$

Given a Beltrami differential $\mu \in \mathcal{B}^{-1,1}(\mathbb{D}, \Gamma)$, extend it to $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ by reflection:

$$(2.1) \quad \mu(z) = \overline{\mu\left(\frac{1}{\bar{z}}\right)} \frac{z^2}{\bar{z}^2}, \quad z \in \mathbb{D}^*.$$

There is a unique quasiconformal mapping $w_\mu : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ which fixes the points $-1, -i, 1$ and satisfies the Beltrami equation $(w_\mu)_z = \mu(w_\mu)_z$. The conjugation of Γ by w_μ , $\Gamma_\mu = w_\mu \circ \Gamma \circ w_\mu^{-1}$, is again a Fuchsian group. The corresponding quotient surface $X_\mu = \Gamma_\mu \backslash \mathbb{D}$ is a Riemann surface having the same type as X , but with different complex structure. Define an equivalence relation on $\mathcal{B}^{-1,1}(\mathbb{D}, \Gamma)$ so that $\mu \sim \nu$ if and only if $w_\mu = w_\nu$ on the unit circle S^1 . Real analytically, the Teichmüller space $T(X)$ is isomorphic to $\mathcal{B}^{-1,1}(\mathbb{D}, \Gamma) / \sim$.

Denote by w^μ the corresponding quasiconformal mapping if we extend $\mu \in \mathcal{B}^{-1,1}(\mathbb{D}, \Gamma)$ to $\hat{\mathbb{C}}$ by setting it equal to zero outside \mathbb{D} . w^μ is holomorphic on \mathbb{D}^* and $\Gamma^\mu = w^\mu \circ \Gamma \circ (w^\mu)^{-1}$ is no longer a Fuchsian group, but a quasi-Fuchsian group. The corresponding Riemann surface $X^\mu = \Gamma^\mu \backslash w^\mu(\mathbb{D})$ is biholomorphic to X_μ . Complex analytically, the Teichmüller space is the quotient $\mathcal{B}^{-1,1}(\mathbb{D}, \Gamma) / \sim$, where $\mu \sim \nu$ if and only if $w^\mu = w^\nu$ on the unit circle. For two compact Riemann surfaces X and Y having the same genus g , their Teichmüller spaces $T(X)$ and $T(Y)$ are naturally isomorphic and we use T_g to denote the Teichmüller space of compact Riemann surfaces of genus g .

The tangent space and cotangent space at a point $[\mu]$ of the Teichmüller space $T(X)$ can be naturally identified with the space of harmonic Beltrami differentials and the space of holomorphic quadratic differentials of X^μ . The space of holomorphic quadratic differentials of X^μ can be identified with the space of Γ_μ -automorphic $(2, 0)$ differentials on \mathbb{D} — $\Omega^{2,0}(\mathbb{D}, \Gamma_\mu)$, which consists of holomorphic functions q on \mathbb{D} satisfying

$$q(\gamma(z))\gamma'(z)^2 = q(z), \quad \forall \gamma \in \Gamma_\mu.$$

Correspondingly, the space of harmonic Beltrami differentials of X^μ can be identified with $\Omega^{-1,1}(\mathbb{D}, \Gamma_\mu)$, which is a subspace of $\mathcal{A}^{-1,1}(\mathbb{D}, \Gamma_\mu)$ consisting of ν of the form

$$\nu(z) = \rho(z)^{-1} \overline{q(z)} = \frac{(1 - |z|^2)^2}{4} \overline{q(z)},$$

where $q \in \Omega^{2,0}(\mathbb{D}, \Gamma_\mu)$ and ρ is the hyperbolic metric density on \mathbb{D} . The moduli space $\mathcal{M}(X)$ is the quotient of the Teichmüller space $T(X)$ under the action of the mapping class group. The Weil–Petersson metric on $T(X)$, which is defined by

$$\langle \nu_\alpha, \nu_\beta \rangle_{WP} = \iint_{\Gamma_\mu \backslash \mathbb{D}} \nu_\alpha(z) \overline{\nu_\beta(z)} \rho(z) d^2 z$$

on the tangent space of $T(X)$ at $[\mu]$, is modular invariant and hence descends to a well-defined metric on the moduli space $\mathcal{M}(X)$.

The universal Teichmüller space $T(1)$ can be defined similarly with the group Γ being the trivial group consisting of only the identity element, i.e., $\Gamma = \{\text{id}\}$. More precisely, $T(1) = \mathcal{B}^{-1,1}(\mathbb{D}) / \sim$, where $\mathcal{B}^{-1,1}(\mathbb{D})$ is the space of bounded functions on \mathbb{D} with sup-norm less than one; and $\mu \sim \nu$ if and only if $w_\mu \sim w_\nu$ on S^1 . The

cotangent space at any point of $T(1)$ is naturally isomorphic to the Banach space

$$A_\infty(\mathbb{D}) = \left\{ q \text{ holomorphic on } \mathbb{D} : \|q\|_\infty = \sup_{z \in \mathbb{D}} \rho(z)^{-1} |q(z)| < \infty \right\};$$

while the tangent space is identified with the Banach space

$$\Omega^{-1,1}(\mathbb{D}) = \left\{ \rho^{-1} \bar{q} : q \in A_\infty(\mathbb{D}) \right\}$$

of harmonic Beltrami differentials on \mathbb{D} . Obviously, the inner product

$$(2.2) \quad \langle \nu_\alpha, \nu_\beta \rangle = \iint_{\mathbb{D}} \nu_\alpha(z) \overline{\nu_\beta(z)} \rho(z) d^2 z$$

is not well-defined on $\Omega^{-1,1}(\mathbb{D})$. In [12], we showed that we can define a Hilbert structure on $T(1)$ so that at any point, its tangent space is isomorphic to

$$H^{-1,1}(\mathbb{D}) = \left\{ \rho^{-1} \bar{q} : q \in A_2(\mathbb{D}) \right\},$$

where

$$A_2(\mathbb{D}) = \left\{ q \text{ holomorphic on } \mathbb{D} : \|q\|_2^2 = \iint_{\mathbb{D}} |q(z)|^2 \rho(z)^{-1} d^2 z < \infty \right\}.$$

We denote the Teichmüller space with this Hilbert structure as $T_H(1)$. The inner product (2.2) is well-defined on the tangent space $H^{-1,1}(\mathbb{D})$ and we called the resulting metric on $T_H(1)$ the Weil–Petersson metric.

Let $\Delta = -\rho^{-1} \partial \bar{\partial}$ be the Laplace–Beltrami operator of the hyperbolic metric on X and let

$$(2.3) \quad G = \frac{1}{2} \left(\Delta + \frac{1}{2} \right)^{-1}$$

be one-half of the resolvent of Δ at $\lambda = -1/2$. In [16], Wolpert showed that the Riemann curvature tensor $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$ of the Weil–Petersson metric at the tangent space of a point on the moduli space correspond to the compact Riemann surface $X = \Gamma \backslash \mathbb{D}$ is given by¹

$$(2.4) \quad R_{\alpha\bar{\beta}\lambda\bar{\delta}} = - \iint_{\Gamma \backslash \mathbb{D}} G(\nu_\alpha \bar{\nu}_\beta)(\nu_\lambda \bar{\nu}_\delta) \rho d^2 z - \iint_{\Gamma \backslash \mathbb{D}} G(\nu_\alpha \bar{\nu}_\delta)(\nu_\lambda \bar{\nu}_\beta) \rho d^2 z.$$

In [12], we generalized this result and showed that formula (2.4) is still valid on $T_H(1)$ if $\Gamma \backslash \mathbb{D}$ is replaced by \mathbb{D} .

3. LOWER BOUNDS OF CURVATURES OF THE WEIL–PETERSSON METRIC ON MODULI SPACE OF COMPACT RIEMANN SURFACES

In [16], Wolpert has shown that the holomorphic sectional, Ricci and scalar curvatures are bounded above by $-\frac{1}{2\pi(g-1)}$, $-\frac{1}{2\pi(g-1)}$ and $-\frac{3(3g-2)}{4\pi}$ respectively. These upper bounds depend on the genus g . On the other hand, Huang showed that the sectional curvature does not have a negative upper bound [3]. Here we would like to find lower bounds for the curvatures, which only depend on the injectivity radius of the corresponding Riemann surface.

¹Our convention differ with the convention of Wolpert by a sign.

Recall that the holomorphic sectional and Ricci curvatures at a point on the moduli space corresponding to the Riemann surface $X = \Gamma \backslash \mathbb{D}$ in the direction spanned by $\nu_\alpha \in \Omega^{-1,1}(\mathbb{D}, \Gamma)$ with $\|\nu_\alpha\|_{WP} = 1$, are given respectively by [2, 16]:

$$(3.1) \quad s_\alpha = R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = -2 \iint_{\Gamma \backslash \mathbb{D}} G(|\nu_\alpha|^2) |\nu_\alpha|^2 \rho d^2 z,$$

and

$$(3.2) \quad \begin{aligned} \mathcal{R}_{\alpha\bar{\alpha}} &= \sum_{\beta=1}^{3g-3} R_{\alpha\bar{\beta}\beta\bar{\alpha}} \\ &= - \sum_{\beta=1}^{3g-3} \left\{ \iint_{\Gamma \backslash \mathbb{D}} G(\nu_\alpha \bar{\nu}_\beta) \bar{\nu}_\alpha \nu_\beta \rho d^2 z + \iint_{\Gamma \backslash \mathbb{D}} G(|\nu_\alpha|^2) |\nu_\beta|^2 \rho d^2 z \right\}, \end{aligned}$$

where $\{\nu_1, \dots, \nu_{3g-3}\}$ is an orthonormal basis of $\Omega^{-1,1}(\mathbb{D}, \Gamma)$. The scalar curvature S is equal to the trace of the Ricci tensor:

$$(3.3) \quad S = \sum_{\alpha=1}^{3g-3} \mathcal{R}_{\alpha\bar{\alpha}} = - \sum_{\alpha=1}^{3g-3} \sum_{\beta=1}^{3g-3} \left\{ \iint_{\Gamma \backslash \mathbb{D}} G(\nu_\alpha \bar{\nu}_\beta) \bar{\nu}_\alpha \nu_\beta \rho d^2 z + \iint_{\Gamma \backslash \mathbb{D}} G(|\nu_\alpha|^2) |\nu_\beta|^2 \rho d^2 z \right\}.$$

On the other hand, given two orthogonal tangent vectors $\nu_\alpha, \nu_\beta \in \Omega^{-1,1}(\mathbb{D}, \Gamma)$ with $\|\nu_\alpha\|_{WP} = \|\nu_\beta\|_{WP} = 1$, the sectional curvature of the plane spanned by the real tangent vectors corresponding to ν_α and ν_β is [2, 16]

$$(3.4) \quad \begin{aligned} K_{\alpha,\beta} &= \frac{1}{4} (R_{\alpha\bar{\beta}\beta\bar{\alpha}} + R_{\beta\bar{\alpha}\alpha\bar{\beta}} - R_{\alpha\bar{\beta}\alpha\bar{\beta}} - R_{\beta\bar{\alpha}\beta\bar{\alpha}}) \\ &= \text{Re} \iint_{\Gamma \backslash \mathbb{D}} G(\nu_\alpha \bar{\nu}_\beta) \nu_\alpha \bar{\nu}_\beta \rho d^2 z - \frac{1}{2} \iint_{\Gamma \backslash \mathbb{D}} G(|\nu_\alpha|^2) |\nu_\beta|^2 \rho d^2 z - \frac{1}{2} \iint_{\Gamma \backslash \mathbb{D}} G(\nu_\alpha \bar{\nu}_\beta) \bar{\nu}_\alpha \nu_\beta \rho d^2 z. \end{aligned}$$

We have used the self-adjointness of G to obtain the last expression.

To obtain the lower bounds of curvatures, Huang [4] used harmonic maps to show that if a harmonic Beltrami differential has unit Weil–Petersson norm, then its sup-norm is bounded above by a constant depending on the injectivity radius of the underlying Riemann surface. Here we reprove this result without resorting to harmonic maps, which better reveals its elementary nature; and also allows us to generalize this result to the universal Teichmüller space later.

Proposition 3.1. *Let $X = \Gamma \backslash \mathbb{D}$ be a compact Riemann surface with injectivity radius r_X and let $\nu \in \Omega^{-1,1}(\mathbb{D}, \Gamma)$ be a harmonic Beltrami differential of X . The ratio of the sup-norm of ν to the Weil–Petersson norm of ν is bounded above by a constant $C(r_X)$ depending only on r_X , i.e.,*

$$\|\nu\|_\infty \leq C(r_X) \|\nu\|_{WP}.$$

The constant $C(r_X)$ can be chosen to be equal to

$$(3.5) \quad C(r_X) = \left\{ \frac{4\pi}{3} \left[1 - \left(\frac{4e^{r_X}}{(e^{r_X} + 1)^2} \right)^3 \right] \right\}^{-\frac{1}{2}}.$$

Proof. Let $z \in \mathbb{D}$ and let

$$\sigma_z(w) = \frac{z + w}{1 + z\bar{w}}, \quad w \in \mathbb{D},$$

be a linear transformation preserving \mathbb{D} and mapping 0 to z . Notice that $\nu \circ \sigma_z$ is a harmonic Beltrami differential of the group $\sigma_z^{-1} \circ \Gamma \circ \sigma_z$, and $\|\nu \circ \sigma_z\|_{WP} = \|\nu\|_{WP}$, but $|\nu(z)| = |\nu \circ \sigma_z(0)|$. Therefore, it suffices to verify that there exists a constant $C(r_X)$ such that

$$|\nu(0)| \leq C(r_X) \|\nu\|_{WP}.$$

By definition, there exists $q \in \Omega^{2,0}(\mathbb{D}, \Gamma)$ such that $\nu = \rho^{-1}\bar{q}$. Being a holomorphic function on \mathbb{D} , q has a Taylor series expansion on \mathbb{D} which can be written as

$$q(z) = \sum_{n=2}^{\infty} (n^3 - n) a_n z^{n-2}.$$

This implies that

$$\nu(0) = \rho(0)^{-1} q(0) = \frac{3a_2}{2},$$

whereas

$$\|\nu\|_{WP}^2 = \iint_{\Gamma \backslash \mathbb{D}} |q(z)|^2 \rho(z)^{-1} d^2 z.$$

By the definition of injectivity radius, we can choose a fundamental domain F for the action of Γ on \mathbb{D} which contains a hyperbolic disc $D(0, r)$ with center at 0 and with radius r , for any r less than r_X . Elementary hyperbolic geometry gives us

$$D(0, r) = \left\{ z \in \mathbb{C} : |z| < \frac{e^r - 1}{e^r + 1} \right\}.$$

Therefore, for any $r \in (0, r_X)$, we have

$$\begin{aligned} \|\nu\|_{WP}^2 &= \iint_F |q(z)|^2 \rho(z)^{-1} d^2 z \\ &\geq \iint_{D(0, r)} |q(z)|^2 \rho(z)^{-1} d^2 z \\ &= \int_0^{\frac{e^r - 1}{e^r + 1}} \int_0^{2\pi} \left| \sum_{n=2}^{\infty} (n^3 - n) a_n u^{n-2} e^{i(n-2)\theta} \right|^2 \frac{(1 - u^2)^2}{4} d\theta u du \\ &= 2\pi \int_0^{\frac{e^r - 1}{e^r + 1}} \sum_{n=2}^{\infty} (n^3 - n)^2 |a_n|^2 u^{2n-4} \frac{(1 - u^2)^2}{4} u du \\ &\geq 18\pi |a_2|^2 \int_0^{\frac{e^r - 1}{e^r + 1}} (1 - u^2)^2 u du \\ (3.6) \quad &= \frac{4\pi}{3} |\nu(0)|^2 \left\{ 1 - \left(\frac{4e^r}{(e^r + 1)^2} \right)^3 \right\}. \end{aligned}$$

Since this is true for all $r \in (0, r_X)$, we can replace r in (3.6) by r_X . Therefore, we have proved the proposition with $C(r_X)$ equal to

$$C(r_X) = \left\{ \frac{4\pi}{3} \left[1 - \left(\frac{4e^{r_X}}{(e^{r_X} + 1)^2} \right)^3 \right] \right\}^{-\frac{1}{2}}.$$

□

Notice that $C(r_X)$ is a decreasing function of r_X . As r_X approaches infinity, it approaches $\sqrt{3/4\pi}$. On the other hand, as $r_X \rightarrow 0$, it behaves like

$$(3.7) \quad C(r_X) \sim \frac{1}{\sqrt{\pi}r_X} + O(1).$$

Before computing the lower bounds for the curvature, we state a useful lemma here.

Lemma 3.2. *Let G be the positive self-adjoint operator on X defined by (2.3).*

A. For any $f \in L^2(X, \mathbb{R})$

$$\iint_{\Gamma \backslash \mathbb{D}} G(f) \bar{f} \rho d^2 z \geq 0.$$

B. For any $f \in L^2(X, \mathbb{R})$,

$$\iint_{\Gamma \backslash \mathbb{D}} G(f) \rho d^2 z = \iint_{\Gamma \backslash \mathbb{D}} f \rho d^2 z.$$

C. If $f \in L^2(X, \mathbb{R})$ is such that $f \geq 0$, then $G(f) \geq 0$.

D. For any $f, g \in L^2(X, \mathbb{R})$, $|G(fg)| \leq G(f^2)^{1/2} G(g^2)^{1/2}$.

Proof. A is an immediate consequence of the positivity of G . B is proved using the self-adjointness of G and the fact that $G(1) = 1$. C follows from the fact that the kernel of $G - G(z, w)$, is a positive function for all z and w (see [12]). D is the Lemma 4.3 in [16]. □

Proposition 3.3. *Let $X = \Gamma \backslash \mathbb{D}$ be a compact Riemann surface of genus g with injectivity radius r_X . At the point on the moduli space \mathcal{M}_g corresponding to X , the holomorphic sectional and sectional curvatures of the Weil-Petersson metric is bounded below by $-2C(r_X)^2$.*

Proof. The proof follows closely the proofs to obtain lower and upper bounds given in [16, 4]. For completeness, we repeat it here. We consider the holomorphic sectional curvature first. Given $\nu_\alpha \in \Omega^{-1,1}(\mathbb{D}, \Gamma)$ with $\|\nu_\alpha\|_{WP} = 1$, we find from (3.1), Proposition 3.1 and B and C of Lemma 3.2 that

$$\begin{aligned} -s_\alpha &\leq 2C(r_X)^2 \iint_{\Gamma \backslash \mathbb{D}} G(|\nu_\alpha|^2) \rho d^2 z \\ &= 2C(r_X)^2 \iint_{\Gamma \backslash \mathbb{D}} |\nu_\alpha|^2 \rho d^2 z = 2C(r_X)^2. \end{aligned}$$

This proves the statement for holomorphic sectional curvature. For the sectional curvature, we use formula (3.4). Notice that Cauchy–Schwarz inequality, the positivity of the kernel of G , and D of Lemma 3.2 give us

$$\begin{aligned}
\left| \iint_{\Gamma \setminus \mathbb{D}} G(\nu_\alpha \bar{\nu}_\beta) \nu_\alpha \bar{\nu}_\beta \rho d^2 z \right| &\leq \iint_{\Gamma \setminus \mathbb{D}} G(|\nu_\alpha| |\nu_\beta|) |\nu_\alpha| |\nu_\beta| \rho d^2 z \\
&\leq \iint_{\Gamma \setminus \mathbb{D}} G(|\nu_\alpha|^2)^{1/2} G(|\nu_\beta|^2)^{1/2} |\nu_\alpha| |\nu_\beta| \rho d^2 z \\
&\leq \left\{ \iint_{\Gamma \setminus \mathbb{D}} G(|\nu_\alpha|^2) |\nu_\beta|^2 \rho d^2 z \right\}^{1/2} \left\{ \iint_{\Gamma \setminus \mathbb{D}} G(|\nu_\beta|^2) |\nu_\alpha|^2 \rho d^2 z \right\}^{1/2} \\
&= \iint_{\Gamma \setminus \mathbb{D}} G(|\nu_\alpha|^2) |\nu_\beta|^2 \rho d^2 z.
\end{aligned}$$

Similarly,

$$(3.8) \quad \left| \iint_{\Gamma \setminus \mathbb{D}} G(\nu_\alpha \bar{\nu}_\beta) \bar{\nu}_\alpha \nu_\beta \rho d^2 z \right| \leq \iint_{\Gamma \setminus \mathbb{D}} G(|\nu_\alpha|^2) |\nu_\beta|^2 \rho d^2 z.$$

Therefore,

$$K_{\alpha, \beta} \geq -2 \iint_{\Gamma \setminus \mathbb{D}} G(|\nu_\alpha|^2) |\nu_\beta|^2 \rho d^2 z.$$

The same reasoning as in the case of holomorphic sectional curvature shows that

$$K_{\alpha, \beta} \geq -2C(r_X)^2.$$

□

If we naively use the approach above to find the lower bounds for the Ricci and scalar curvatures, we will find that the Ricci and scalar curvatures are bounded below by $-2(3g-3)C(r_X)^2$ and $-2(3g-3)^2 C(r_X)^2$ respectively. In fact, these bounds are obtained in [4]. However, by doing slightly more work, we can greatly improve the bounds. Observe that given an orthonormal basis $\{\nu_1, \dots, \nu_{3g-3}\}$ of $\Omega^{-1,1}(\mathbb{D}, \Gamma)$, the kernel

$$(3.9) \quad P(z, w) = \rho(z) \rho(w) \sum_{\beta=1}^{3g-3} \overline{\nu_\beta(z)} \nu_\beta(w)$$

is the kernel of the projection operator mapping bounded quadratic differentials to holomorphic quadratic differentials. Namely, for any bounded quadratic differential q of X ,

$$(Pq)(z) := \iint_{\Gamma \setminus \mathbb{D}} P(z, w) q(w) \rho(w)^{-1} d^2 w$$

is a holomorphic quadratic differential, and $Pq = q$ if and only if q is holomorphic. Let

$$(3.10) \quad \Lambda = \sup_{z \in \mathbb{D}} \sum_{\beta=1}^{3g-3} |\nu_\beta(z)|^2 = \sup_{z \in \mathbb{D}} \rho(z)^{-2} P(z, z).$$

Obviously,

$$(3.11) \quad \Lambda \leq \sup_{z \in \mathbb{D}} \sup_{w \in \mathbb{D}} \left| \sum_{\beta=1}^{3g-3} \overline{\nu_\beta(z)} \nu_\beta(w) \right|.$$

For fixed z , $\sum_{\beta=1}^{3g-3} \overline{\nu_\beta(z)} \nu_\beta(w) \in \Omega^{-1,1}(\mathbb{D}, \Gamma)$. By (3.9), its Weil–Petersson norm is

$$\left\| \sum_{\beta=1}^{3g-3} \overline{\nu_\beta(z)} \nu_\beta(w) \right\|_{WP}^2 = \rho(z)^{-2} \iint_{\Gamma \backslash \mathbb{D}} P(z, w) \overline{P(z, w)} \rho(w)^{-1} d^2 w.$$

Since as a function of w , $\overline{P(z, w)} = P(w, z)$ is a holomorphic quadratic differential, the projection property of P implies that

$$\iint_{\Gamma \backslash \mathbb{D}} P(z, w) \overline{P(z, w)} \rho(w)^{-1} d^2 w = \overline{P(z, z)} = P(z, z).$$

Therefore,

$$\left\| \sum_{\beta=1}^{3g-3} \overline{\nu_\beta(z)} \nu_\beta(w) \right\|_{WP}^2 = \rho(z)^{-2} P(z, z).$$

Using Proposition 3.1, this gives

$$\sup_{w \in \mathbb{D}} \left| \sum_{\beta=1}^{3g-3} \overline{\nu_\beta(z)} \nu_\beta(w) \right| \leq C(r_X) \sqrt{\rho(z)^{-2} P(z, z)}.$$

Consequently, (3.11) and (3.10) imply that

$$\Lambda \leq \sup_{z \in \mathbb{D}} \sup_{w \in \mathbb{D}} \left| \sum_{\beta=1}^{3g-3} \overline{\nu_\beta(z)} \nu_\beta(w) \right| \leq C(r_X) \sup_{z \in \mathbb{D}} \sqrt{\rho(z)^{-2} P(z, z)} = C(r_X) \Lambda^{1/2}.$$

In other words,

$$(3.12) \quad \Lambda = \sup_{z \in \mathbb{D}} \sum_{\beta=1}^{3g-3} |\nu_\beta(z)|^2 \leq C(r_X)^2.$$

Notice that we greatly improve the naive bound $\Lambda \leq (3g-3)C(r_X)^2$ to $\Lambda \leq C(r_X)^2$. Now we can prove the following lower bounds for Ricci and scalar curvatures:

Proposition 3.4.

- A. *The Ricci curvature of the Weil–Petersson metric is bounded below by $-2C(r_X)^2$.*
- B. *The scalar curvature of the Weil–Petersson metric is bounded below by $-2(3g-3)C(r_X)^2$.*

Proof. Using (3.2), (3.3) and (3.8), we have

$$\begin{aligned}\mathcal{R}_{\alpha\bar{\alpha}} &\geq -2 \sum_{\beta=1}^{3g-3} \int_{\Gamma \setminus \mathbb{D}} G(|\nu_\alpha|^2) |\nu_\beta|^2 \rho d^2 z, \\ S &\geq -2 \sum_{\alpha=1}^{3g-3} \sum_{\beta=1}^{3g-3} \int_{\Gamma \setminus \mathbb{D}} G(|\nu_\alpha|^2) |\nu_\beta|^2 \rho d^2 z.\end{aligned}$$

Eq. (3.12) and the same method used in the proof of Proposition 3.3 then give us immediately

$$\mathcal{R}_{\alpha\bar{\alpha}} \geq -2C(r_X)^2, \quad S \geq -2(3g-3)C(r_X)^2.$$

□

Notice that we have established that the Ricci curvature of the Weil–Petersson metric is bounded below by a constant depending only on the injectivity radius of the corresponding Riemann surface, but independent of the genus. This substantially improves the result of [4].

We would also like to remark that tending to the boundary of the moduli spaces, the injectivity radius $r_X = \text{inj}(X)$ decreases to zero. The results of Propositions 3.3 and 3.4 and the estimate (3.7) show that when $r_X \rightarrow 0$, the holomorphic sectional, sectional and Ricci curvatures of the Weil–Petersson metric are all bounded below by a constant of order $1/r_X^2$. It is interesting to compare this with the asymptotics of the curvatures obtained in [11].

4. BOUNDS OF CURVATURES ON THE UNIVERSAL TEICHMÜLLER SPACE

In [12], we have shown that the holomorphic sectional and sectional curvatures of the Weil–Petersson metric on the Hilbert manifold $T_H(1)$ are negative. We also showed that $T_H(1)$ is a Kahler–Einstein manifold with constant Ricci curvature $-\frac{13}{12\pi}$. In this section, we show that the holomorphic sectional and sectional curvatures are bounded below by a universal constant. We also show that these curvatures do not have negative upper bounds.

An analog of Proposition 3.1 for the universal Teichmüller space $T_H(1)$ is

Lemma 4.1. *Let $\nu \in \Omega^{-1,1}(\mathbb{D})$ be a harmonic Beltrami differential on \mathbb{D} . Then*

$$(4.1) \quad \|\nu\|_\infty \leq \sqrt{\frac{3}{4\pi}} \|\nu\|_{WP}.$$

This can be considered as the limiting case of Proposition 3.1 when $r_X \rightarrow \infty$. In fact, in the present situation, the corresponding Riemann surface is isomorphic to the disc which has infinite hyperbolic radius. Another proof of (4.1) is given in the proof of Lemma 2.1 in [12].

Using Lemma 4.1, one obtains immediately as in Proposition 3.3 that

Proposition 4.2. *On the universal Teichmüller space $T_H(1)$, the holomorphic sectional and sectional curvatures are bounded below by $-\frac{3}{2\pi}$.*

To prove the statements about upper bounds, we define for $n \geq 2$,

$$\nu_n = \rho(z)^{-1} \sqrt{\frac{2(n^3 - n)}{\pi}} \bar{z}^{n-2}.$$

It is easy to show that $\{\nu_2, \nu_3, \dots\}$ is an orthonormal basis of $H^{-1,1}(\mathbb{D})$. It is elementary to find the sup-norm of ν_n explicitly:

Lemma 4.3. *For $n \geq 2$,*

$$\|\nu_n\|_\infty = \sqrt{\frac{2(n^3 - n)}{\pi}} \frac{4}{(n+2)^2} \left(\frac{n-2}{n+2}\right)^{\frac{n-2}{2}}.$$

Proof. For $n \geq 2$, define

$$h_n(r) = (1 - r^2)^2 r^{n-2}, \quad r \in [0, 1].$$

Then h_n is a nonnegative function and

$$h'_n(r) = r^{n-3}(1 - r^2)((n-2) - (n+2)r^2).$$

This implies that $h_n(r)$ has maximum at $r = \sqrt{(n-2)/(n+2)}$ and its maximum value is

$$\max_{r \in [0,1]} h_n(r) = \frac{16}{(n+2)^2} \left(\frac{n-2}{n+2}\right)^{\frac{n-2}{2}}.$$

The assertion follows. \square

Notice that $\|\nu_2\|_\infty = \sqrt{3/(4\pi)}$. This shows that the result of Lemma 4.1 is sharp. On the other hand, it is easy to see that

$$(4.2) \quad \|\nu_n\|_\infty \leq \sqrt{\frac{32}{\pi(n+2)}} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Using this, we can prove that

Proposition 4.4. *On the universal Teichmüller space $T_H(1)$, the holomorphic sectional and sectional curvatures do not have negative upper bounds.*

Proof. For the holomorphic sectional curvature, we obtain as in the proof of Proposition 3.3 that

$$|s_n| = 2 \iint_{\mathbb{D}} G(|\nu_n|^2) |\nu_n|^2 \rho d^2 z \leq 2 \|\nu_n\|_\infty^2.$$

On the other hand, the proof of Proposition 3.3 shows that the sectional curvature $K_{m,n}$ (3.4) is bounded by

$$|K_{m,n}| \leq 2 \iint_{\mathbb{D}} G(|\nu_m|^2) |\nu_n|^2 \rho d^2 z \leq 2 \|\nu_n\|_\infty^2.$$

Since by (4.2), $\|\nu_n\|_\infty \rightarrow 0$ as $n \rightarrow \infty$, we conclude that the holomorphic sectional and sectional curvatures do not have negative upper bounds. \square

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